Quantum Entanglement

After a long break, I finally had the time to write up another short article about an interesting in my opinion phenomenon. Quantum entanglement is a known fact, but somehow to a lot of people outside the field of physics (and sadly, to some people in the field) it remains as something mystical and unstudied. Without going into philosophy, I decided to briefly derive the mathematical principles leading to entanglement and quickly summarize the idea behind it. This article should be comprehensible to everyone with basic knowledge of linear algebra and quantum mechanics. Come to think of it, prior knowledge of quantum mechanics is not really a must, provided that the reader has a firm grasp of what the term "state" stands for. This is usually provided in more general courses, for example an introductory course in atomic physics. Let us begin:

Entanglement of pure states

Firstly, a Definition 1 A Pure quantum state is such state, that is expressed as a vector in a complex Hilbert space and has unit length. That means that for the pure state $|\psi\rangle$ and an arbitrary basis $|u_1\rangle, \cdots, |u_n\rangle$, the state $|\psi\rangle$ can be expressed as

$$|\psi\rangle = \alpha_1|u_1\rangle + \cdots + \alpha_n|u_n\rangle$$

where $\sum_{i=1}^{n}|\alpha_i|^2 = 1$. Let us now clarify what exactly we call entanglement in the case of pure states. Let $|u_1\rangle, \cdots, |u_n\rangle$ and $|v_1\rangle, \cdots, |v_m\rangle$ be orthonormal bases in the n and m dimensional Hilbert spaces H_n and H_m respectively. We denote by $H_{composite}$ the direct product of the spaces H_n H_m. Constructed in this way, $H_{composite}$ is a $n \cdot m$ dimensional Hilbert space with an orthonormal basis

$$|u_i\rangle \otimes |v_j\rangle \quad i = 1, 2, \cdots, n, j = 1, 2, \cdots, m,$$

where $|u_i\rangle \otimes |v_j\rangle = |u_i\rangle|v_j\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m}\gamma_{ij}|u_i\rangle|v_j\rangle$ and $\sum_{i=1}^{n}\sum_{j=1}^{m}|\gamma_{ij}|^2 = 1$.

Definition 2: A pure state $|\chi\rangle \in H_{nm}$ is called separable if and only if it can be written as a tensor product of states $|\psi\rangle = \sum_{i=1}^{n}\alpha_i|u_i\rangle \in H_n$ and $|\varphi\rangle = \sum_{j=1}^{n}\beta_j|u_j\rangle \in H_m$, meaning that the state $|\chi\rangle$ looks like

$$|\chi\rangle = |\psi\rangle \otimes |\varphi\rangle$$

If a state is not separable, it's called entangled.

Let us clarify the situation with an example.

Consider the state $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Obviously, it is a pure state (you can show that, the calculation is trivial!) and belongs to $H_2 \otimes H_2$. Let $|\varphi_1\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\varphi_2\rangle = \gamma|0\rangle + \delta|1\rangle$ be states in the Hilbert spaceH_2 and $|\alpha|^2 + |\beta|^2 = 1$ and $|\gamma|^2 + |\delta|^2 = 1$. Suppose that $|\psi^+\rangle$ is separable, meaning that it can be written as:

$$|\psi^+\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle).$$

From this it follows that

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle.$$

This means that both $\alpha\gamma = \beta\delta = \frac{1}{\sqrt{2}}$ and $\alpha\delta = \beta\gamma = 0$, which is a contradiction. A contradiction means that our proposition was wrong! So, in fact $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is an entangled state.

Entanglement of mixed states

Every pure state $|\psi\rangle$ can be equated to a density matrix operator, defined as $\rho = |\psi\rangle\langle\psi|$. Mixed states are a statistical "mixture" of density matrices of pure states. Every density matrix, ρ, is a projection operator to a one-dimensional space, meaning that it satisfies the equation $\rho^2 = \rho$.

Let us consider a statistical mix of pure states $\rho_i = |\psi_i\rangle\langle\psi_i|, i = 1, 2, \cdots, n$ with probabilities $\mathrm{p}_i, i = 1, 2, \cdots, n$. Then the density matrix operator of the states can be written as

$$\rho = \sum_{i=1}^{n} p_i \rho_i \quad (1)$$

Note The density matrix has the following important properties – it is Hermitian, positive and $Tr(\rho) = 1$.

Definition 3: Let H_A and H_B be Hilbert spaces. We denote the density matrix of the states in $H_A \otimes H_B$ with ρ. The operator ρ is called separable if there exists a series $(\rho_i)_{i=1}^n$ of positive real number, adding up to one, a series of density matrices $(\rho_i^A)_{i=1}^n$, corresponding to the states in H_A and a series of density matrices $\left(\rho_i^B\right)_{i=1}^n$ corresponding to the states in H_B such that,

$$\rho = \sum_{i=1}^{n} p_i \rho_i^A \otimes \rho_i^B \quad (1)$$

Boy, that was a mouthful! Let us paraphrase that a bit, to make it more comprehensible. If a mixed state can be written as a convex combination of direct products of density matrices, then it is separable. Equation (1) is more restrictive than the famous Bell inequalities (of which I may at some point write a post but here's a link to Wikipedia), therefore each separable state satisfies the Bell inequalities. (1) also applies for pure states. Sadly, the state ρ given by (1) is not unique. Figuring out if a state is entangled is a hard question, except in the case of pure states, where we can use the process from the example given earlier. There are developed procedures that can determine if a mixed state is entangled, but they are not universal – they work only in special cases and are far beyond the scope of this article.

Entropy of mixed states

In information theory, the Shannon entropy (also known as the Jenses-Shannon divergence) is a measure for the expected information, contained in a message. The quantum analog of that measure is the Von Neumann entropy. In the classical world, the entropy of a random variable is never bigger than that of coupled random variables. In the quantum case, however, there are cases when the entropy of a system may be less than the sum of the entropies of its subsystem.

Definition 4 Let ρ be a density matrix of a quantum system. The the Von Neumann entropy is given by

$$S(\rho) = -\mathrm{tr}(\rho \log \rho).$$

Using spectral decomposition, the logarithm can be extended to operators

$$S(\rho) = -\sum_{i=1}^{n} (\lambda_i \log \lambda_i).$$

Here and thereafter, the logarithms have base 2.

Every pure state has a spectrum of the type $\lambda_1 = 1, \lambda_2 = 0, \cdots, \lambda_n = 0$. That means that the Von Neumann entropy of a system of pure states is equal to

$$S(\rho) = -1 \log 1 = 0$$

Definition 5: A mixed state is called maximally mixed if it's represented by a density matrix $\rho = \frac{1}{N} 1 \in H$, where N is the dimension of the space H. The entropy of these states takes a maximum value and is equal to

$$S(\rho) = -\sum_{i=1}^{n} \frac{1}{n} \log \frac{1}{n} = \log n.$$

This means that the Von Neumann entropy can be thought of as a measure of uncertainty of a quantum state measurement.

We're almost at the final example and the conclusion of this post. Before that, however, we need to define what we mean by partial trace. For each separable state $\rho^{AB} = \rho^A \otimes \rho^B$, the partial traces are defined as following

$$\mathrm{tr}_A(\rho^{AB}) = \mathrm{tr}(\rho^A)\rho^B \quad and \quad \mathrm{tr}_B(\rho^{AB}) = \mathrm{tr}(\rho^B)\rho^A.$$

Because ρ^A and ρ^B are density matrices, they have trace equal to one. Then the reduced density matrices can be expressed in terms of the partial traces:

$$\rho^A = \mathrm{tr}_B(\rho^{AB}) \quad and \quad \rho^B = \mathrm{tr}_A(\rho^{AB})$$

Since taking the trace is a linear operation, this is true for all states.

Definition 6: Let ρ^A, ρ^B and ρ^{AB} be the density matrices of the quantum systems $\mathcal{A}$, $\mathcal{B}$ and the composite quantum system $\mathcal{AB}$, respectively. The total Von Neumann entropy of the systems $\mathcal{A}$ and $\mathcal{B}$ is defined as

$$S(\rho^A, \rho^B) = S(\rho^{AB}).$$

Definition 7: Let ρ^A, ρ^B be the density matrices of the quantum systems $\mathcal{A}$, $\mathcal{B}$. The conditional Von Neumann entropy is defined as

$$S(\rho^A|\rho^B) = S(\rho^A, \rho^B) - S(\rho^B).$$

We can now see that the quantum conditional entropy can be negative, which is not possible in the classical case.

Ok! This got a bit technical and maths-y which, depending on your preference might be either very good or very boring. So to spice things up and to summarize and conclude the whole post, I'll give a final example. I urge you to read and follow it carefully, because it's a great insight on how all of this machinery works.

Final Example (Huzzah!)

Let us consider an entangled state $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ in the system $\mathcal{AB}$. Writing the density matrix ρ^{AB} for the state $|\psi^+\rangle$ we have

$$\rho^{AB} = \begin{pmatrix} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \end{pmatrix}$$

The spectrum of this operator is $(1, 0, 0, 0)$, therefore the conditional entropy of the state ρ^{AB} is

$$S(\rho^A|\rho^B) = -\log 1 = 0$$

The reader may want to try to prove that

$$\rho^A = \rho^B = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|).$$

Then the spectra of the operators ρ^A and ρ^B is the set $(\frac{1}{2}, \frac{1}{2})$. From here, we can calculate the entropy of the operators ρ^A and ρ^B. We get

$$S(\rho^A) = S(\rho^B) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = 1.$$

But what does this mean? Reading carefully, we just proved mathematically the inequality $S(\rho^A) \geq S(\rho^{AB})$. This means that the entropy of the subsystem is greater than the entropy of the composite system! In layman terms (but without losing any rigor), this means that knowing everything about the subsystem is not enough to describe the composite system. This property only manifests itself when we deal with entangled states.

In conclusion, quantum entanglement is a physical phenomenon, which manifests when two or more particles interact in such a way that the quantum states of the particles cannot be described independently from one another. In those case, the whole system is described by one entangled state. This has an effect of the entropy, meaning if affects the amount of information needed for describing the system. The phenomenon is described using the language of linear algebra, via non-diagonal density matrices. Here our brief overview of entanglement ends. If you want me to write up a post describing something which you have an interest in, or you have noticed some mistakes, please leave your message in the comments.